

General Relativity Week 10

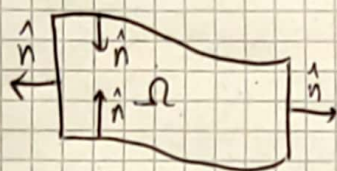
Last time: We saw that if ψ solves

$$\square_g \psi = F, \quad \text{setting } T_{\mu\nu}[\psi] = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} \partial^\alpha \psi \partial_\alpha \psi g_{\mu\nu}$$

then $\boxed{\operatorname{div} J^{(X)}[\psi] = F \cdot X(\psi) + T_{\mu\nu}[\psi] \nabla^\mu X^\nu}$

(Where $J_\mu^{(X)}[\psi] = T_{\mu\nu}[\psi] \cdot X^\nu$) $\frac{1}{2}(\nabla^\mu X^\mu + \nabla^\nu X^\nu) = \frac{1}{2}(\nabla_\mu X^\mu)$

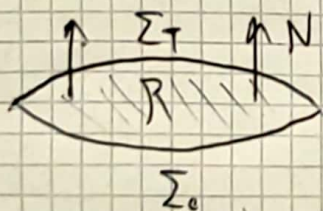
And: Divergence theorem: If $\Omega \subseteq (M, g)$ has only spacelike or null components for its boundary, then $\forall w \in \Gamma^*(M)$:



$$\int_{\partial\Omega} \operatorname{div} w \cdot d\text{vol}_g = \int_{\partial\Omega} w(\hat{n}) d\text{vol}_g$$

We will use the above to get an energy-type inequality on general spacetimes.

Let R be a domain of a spacetime (M, g) that looks like a "lens", i.e.



- $\bar{R}$ is compact $\rightarrow$ level set of function τ
- $R = \bigcup_{\tau \in (0, T)} \Sigma_\tau$, Σ_τ : strictly spacelike
- $\partial R = \Sigma_0 \cup \Sigma_T$
- $\exists$ timelike vector field N defined on a neighborhood of $\bar{R}$

Note that with these assumptions: (R, g) is globally hyperbolic, each Σ_τ is a Cauchy hypersurface.

The "time" function τ is smooth in the interior of R ,
but singular at the corners. It satisfies $\sup_R \frac{1}{|\nabla \tau|} \leq C$.

Let $\Box_g \psi = F$ on a neighborhood of $\bar{R}$.

Integrate over R the identity

$$\operatorname{div} J^{(N)} = F \cdot N(\psi) + T_{\mu\nu} \cdot \nabla^\mu N^\nu$$

$\Downarrow$ divergence theorem

$$\textcircled{A} \quad \int_{\Sigma_T} J_\mu^{(N)} \hat{n}^\mu = \int_{\Sigma_0} J_\mu^{(N)} \hat{n}^\mu + \int_R T_{\mu\nu}[\psi] \nabla^\mu N^\nu \, d\text{vol}_g \quad \textcircled{A}$$

$$+ \int_R F \cdot N\psi \, d\text{vol}_g \quad \textcircled{B}$$

here: We fix $\hat{n}$ to be future directed
unit normal to Σ_τ

Let us set $f(\tau) = \int_{\Sigma_\tau} J_\mu^{(N)}[\psi] \hat{n}^\mu$

Since $N, \hat{n}$ timelike and future directed:

In any local coordinate chart $f(\tau) \geq c \int_{\Sigma_\tau \cap \{\text{coordinate chart}\}} |\partial \psi|^2$

Because of compactness of $\bar{R}$:
I can choose this constant uniformly.

For the term $\textcircled{A}, \textcircled{B}$: Co-area formula: $\int_R h \, d\text{vol}_g = \int_0^T \int_{\Sigma_\tau} h \cdot \frac{1}{|\nabla \tau|} \, d\text{vol}_g$

Combining the co-area formula with the Cauchy-Schwarz inequality:

$$\textcircled{A} = \int_R T_{\mu\nu}[\psi] \nabla^\mu N^\nu = \int_0^T \int_{\Sigma_\tau} T_{\mu\nu}[\psi] \cdot \nabla^\mu N^\nu \cdot \frac{1}{|\nabla \tau|} d\text{vol}_{\bar{g}} d\tau$$

In local coordinates: $|T_{\mu\nu}[\psi]| \lesssim |\partial\psi|^2$

$$|\nabla^\mu N^\nu \cdot \frac{1}{|\nabla \tau|}| \lesssim 1$$

$$\leq C \int_0^T f(\tau) d\tau$$

And $\textcircled{B} = \int_R F \cdot N\psi \leq \left(\int_R F^2\right)^{1/2} \left(\int_R (N\psi)^2\right)^{1/2}$
 $\leq C \left(\int_R F^2\right)^{1/2} \left(\int_0^T f(\tau) d\tau\right)^{1/2} \leq C \int_R F^2 + C \int_0^T f(\tau) d\tau$

So $\textcircled{A} \Rightarrow f(T) \leq f(0) + C \int_0^T f(\tau) d\tau + C \int_R F^2$

Gronwall's inequality

$\Rightarrow$

$$\sup_{\tau \in [0, T]} f(\tau) \leq C' f(0) + C' \int_R F^2$$

C' : Depends on R, g, N but not on ψ !

• In other words:

Energy inequality

$$\sup_{\tau \in [0, T]} \|\partial\psi\|_{L^2(\Sigma_\tau)}^2 \leq C' \left(\|\partial\psi\|_{L^2(\Sigma_0)}^2 + \|F\|_{L^2(R)}^2 \right)$$

We can also "improve" the above energy estimate as follows:

Let us set

$$\mathcal{E}^{(1)}[\psi](\tau) := \int_{\Sigma_\tau} \left(\bigvee_{\mu}^{(N)} [\psi] \hat{n}^\mu + \psi^2 \right) d\text{vol}_{\bar{g}}$$

$$\left(\text{So } \mathcal{E}^{(1)}[\psi](\tau) = \|\psi\|_{H^1(\Sigma_\tau)}^2 + \|\hat{n}(\psi)\|_{L^2(\Sigma_\tau)}^2 \right)$$

^ only tangential derivative to Σ_τ

then

$$\sup_{\tau \in [0, T]} \mathcal{E}^{(1)}[\psi](\tau) \lesssim \mathcal{E}^{(1)}[\psi](0) + \|F\|_{L^2(R)}^2$$

Proof: Given the previous energy identity:

It suffices to ~~also~~ estimate $\sup_{\tau \in [0, T]} \int_{\Sigma_\tau} \psi^2 d\text{vol}_{\bar{g}}$.

In any local coordinate system of the form $(\tau, x^1, \dots, x^n)$ on a domain U which is τ -invariant:

$$\int_{\Sigma_\tau \cap U} \psi^2 d\text{vol}_{\bar{g}} = \int_{\Sigma_0 \cap U} \psi^2 d\text{vol}_{\bar{g}} + \int_0^\tau \int_{\Sigma_s \cap U} \underbrace{\partial_\tau(\psi^2)}_{2\partial_\tau \psi \cdot \psi} d\text{vol}_{\bar{g}} ds$$

$$\leq \int_{\Sigma_0 \cap U} \psi^2 + \int_0^\tau \left(\int_{\Sigma_s \cap U} (\partial_\tau \psi)^2 d\text{vol}_{\bar{g}} \right) ds + \int_0^\tau \left(\int_{\Sigma_s \cap U} \psi^2 d\text{vol}_{\bar{g}} \right) ds$$

This term: $\lesssim \int_{\Sigma_s} \bigvee_{\mu}^{(N)} \hat{n}^\mu$ (Near the corners of R : ∂_τ is "small")

So applying Gronwall ... $\square$

Note: We proved the energy estimate for $\Box_g \psi = F$, ψ real valued.

Exactly the same proof works for

$$\Box_g \psi^k + A_a^k \partial_a \psi^2 + B_a^k \psi^2 = F^k$$

(i.e. vector valued, "diagonal" principal symbol)

(just put the lower order terms on the right hand side and treat them like F)

But then: I can get higher order energy estimates:

$$\text{If } \Box_g \psi + A \partial \psi + B \psi = F$$

$$\text{Then } \forall k: \text{ If } V^{(k)} = \begin{pmatrix} \psi \\ \partial \psi \\ \vdots \\ \partial^k \psi \end{pmatrix} \text{ then}$$

$$\Box_g V^{(k)} + \tilde{A} \partial V^{(k)} + \tilde{B} V^{(k)} = F^{(k)} = \begin{pmatrix} F \\ \partial F \\ \vdots \\ \partial^k F \end{pmatrix}$$

So applying the energy estimate for $V^{(k)}$:

We get

$$\sup_{t \in [0, T]} \mathcal{E}^{(k)}[\psi](t) \lesssim \mathcal{E}^{(k)}[\psi](0) + \|F\|_{H^{k-1}(R)}^2$$

where

$$\mathcal{E}^{(k)}[\psi](t) = \sum_{|a|=0}^k \int_{\Sigma_t} |\partial^a \psi|^2$$

$$\partial^a = \partial_{x_0}^{a_0} \partial_{x_1}^{a_1} \dots \partial_{x_n}^{a_n}$$

$$\text{if } a = (a_0, \dots, a_n)$$

$$(|a| = a_0 + \dots + a_n)$$

(Can give a coordinate-independent formulation)

From this: propagation of regularity from the initial data.

In L^2 -norms

(this is not true in other L^p space, except in 1+1 dimensions)

Let us consider the initial value problem on R :

$$\begin{cases} \Box_g \psi = F \\ \psi|_{\Sigma_0} = \psi_0, \quad \hat{n}(\psi')|_{\Sigma_0} = \psi_1 \end{cases} \quad (1)$$

Applying the energy estimate to the difference of two solutions:

- Uniqueness

- Continuous dependence of ψ on (ψ_0, ψ_1, F)

The map $(\psi_0, \psi_1, F) \longrightarrow \psi$

is continuous from $H^k(\Sigma_0) \times H^{k-1}(\Sigma_0) \times H^{k-1}(R) \longrightarrow L_b^\infty E^{(k)}$

$\uparrow$
 finite
 k -th energy

In order to establish well-posedness for (1): It remains to show the existence of solutions.

Theorem: (1) has a solution ψ on R if (ψ_0, ψ_1, F) are smooth.

Proof: It suffices to show this for $(\psi_0, \psi_1) = (0, 0)$

(In general: Extend (ψ_0, ψ_1) to arbitrary function $\tilde{\psi}$,

then $\psi - \tilde{\psi}$ solves $\Box(\psi - \tilde{\psi}) = F - \Box\tilde{\psi}$ with 0 initial data)

Assume w.l.o.g. that $T=1$.

Let $X = \{ \phi \in C^\infty(R) : \phi, d\phi|_{\Sigma_1} = 0 \}$

We want ψ to solve $\forall \phi \in X$:

$$\int_{\mathbb{R}} \psi \cdot \square_g \phi = \int_{\mathbb{R}} F \cdot \phi \quad (W)$$

$\uparrow$

Weak ("distributional") formulation of the equation:

Makes sense for ψ just a "distribution".

We will show that a $\psi \in L^2(\mathbb{R})$ exists solving (W).

Let $Y = \{ \square_g \phi : \phi \in X \} \subseteq L^2(\mathbb{R})$

Define the functional $L: Y \rightarrow \mathbb{R}$

$$\text{by } L[\square_g \phi] = \int_{\mathbb{R}} F \cdot \phi$$

By the energy estimate: $\sup_{\Sigma_-} \|\phi\|_{L^2(\Sigma_-)} \lesssim \|\square_g \phi\|_{L^2(\mathbb{R})}$

(solving backwards
from Σ_+)

$$\text{So } \|\phi\|_{L^2(\mathbb{R})} \lesssim \|\square_g \phi\|_{L^2(\mathbb{R})}$$

$$\text{Hence: } \left| \int_{\mathbb{R}} F \cdot \phi \right| \lesssim \|F\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \lesssim \|F\|_{L^2(\mathbb{R})} \|\square_g \phi\|_{L^2(\mathbb{R})}$$

$$\Rightarrow |L[\square_g \phi]| \lesssim \|F\|_{L^2(\mathbb{R})} \|\square_g \phi\|_{L^2(\mathbb{R})}$$

So L is bounded on $(Y, \|\cdot\|_{L^2})$

$\Rightarrow$ Hahn-Banach: $\exists$ extension L to $L^2(\mathbb{R})$ with the same norm

By the Riesz Representation

$$\exists \psi \in L^2(\mathbb{R}) \quad \text{s.t.} \quad \mathcal{L}[z] = \int_{\mathbb{R}} \psi \cdot z$$

So ψ solves the weak form of the equation.



One can actually show that ψ is more regular, and thus solves the "strong" version of the equation

(By showing boundedness not in L^2 , but in H^{-k} , $k > 0$)